

Cosmological redshift and nonlinear electrodynamics propagation of photons from distant sources

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By-now photons are the unique universal messengers. Cosmological sources like far-away galaxies or quasars are well-known light-emitters. Here we demonstrate that the nonlinear electrodynamics (NLED) description of photon propagation through the weak background intergalactic magnetic fields modifies in a fundamental way the cosmological redshift that a direct computation within a specific cosmological model can ascribe to a distant source. Independently of the class of NLED Lagrangian, the effective redshift turns out to be $1 + \tilde{z} = (1 + z) \Delta$, where $\Delta \equiv (1 + \Phi_e)/(1 + \Phi_o)$, with $\Phi \equiv 8/3(L_{FF}/L_F)B^2$, being $L_F = dL/dF$, $L_{FF} = d^2L/dF^2$, the field $F \equiv F_{\alpha\beta}F^{\alpha\beta}$, and B the magnetic field strength. Thus the effective redshift is always much higher than the standard redshift, but recovers such limit when the NLED correction $\Delta(\Phi_e, \Phi_o) \rightarrow 1$. This result may provide a physical foundation for the current observation-inspired interpretation that the universe undergoes an accelerate expansion. However, under the situation analyzed here, for any NLED the actual (spatial) position of the light-emitting far-away source remains untouched.

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Introduction.— The study of the expansion history of the universe gained a novel dimension after the discovery of what appears to be a dimming in the luminosity emitted by supernovae type Ia (SNIa) [1], which are thought of as standard candles. These observations have been interpreted in the context of the standard cosmological model as evidence of a late-time transition from decelerate-to-accelerate expansion, according to most current viewpoints. The conclusion is attained after combining both the redshift and luminosity-distance of observed SNIa events in their Hubble diagram (HD). If one excludes any potential systematics (as for instance, there exists the possibility that we are being unable to detect, due to dust effects, much more reddened SNIa taking place at much higher redshifts (z), than close-by bluish explosions because simply these late ones are much brighter, see Ref.[2]), one can verify that their representative points in the HD appear a bit over the upper bound curve predicted by the standard Friedmann cosmology, and pile-up around $z \sim 0.5 - 1$, which is referred to as the transition era.

The redshift is the fundamental piece in achieving this conclusion. It is determined in almost all the cases by analysing the absorption lines from SNIa host galaxies. However, the unavoidable nonlinear interaction of light [3] from these distant sources with the intergalactic background magnetic fields [23] may crucially modify the putative value of the redshift to be ascribed to a given source from the observed lines. On the other hand, exception done for the case in which the electric permittivity and magnetic permeability are functions of the fields, i.e., $\epsilon(\vec{E}, \vec{B})$, $\mu(\vec{E}, \vec{B})$, Maxwell electrodynamics is unable to describe the nonlinear behavior of light propagation. If one follows this line of reasoning, one realizes that one way to guide ourselves to a proper investigation of the nonlinear interaction

of photons [3] from distant galaxies and quasars with intergalactic background fields is to keep in mind that those background magnetic fields are extremely weak! (observations rule out any electric fields, i. e., $< E > = 0!$). Hence, if a given Lagrangian will indeed describe such photon nonlinear dynamics, it will have to depend on the invariant $F \equiv F_{\mu\nu}F^{\mu\nu}$ field in a nontrivial fashion. Interestingly, a hint to the need for a nonlinear electrodynamics (NLED) Lagrangian able to account for such dynamics came to us from the study of a correlated phenomenon: claims on a potential variation of the fine structure constant α (see [11] and references therein). Indeed, Murphy et al. [4] (Section 2.6) based on Maxwell electromagnetic theory considered large magnetic fields as a potential cause of systematic errors in their measurements of $\Delta\alpha/\alpha$, and conclude that the intra-cluster magnetic field strengths are nine orders of magnitude below the strength required to cause substantial effects. Therefore, it appears legitimate to address the question of any putative modification of the standard cosmological redshift from far away astrophysical sources within the framework of NLED. One way to achieve this goal is to use the Lagrangian for NLED recently introduced in Ref.[6] (see also Ref.[17]), whose original focus was to bring in an alternative to dark energy to explain the universe late-time accelerate expansion. In passing, it turns to be easy to check that Murphy et al.'s conclusion can be reversed by considering the NLED Lagrangian density of Ref.[6].

Nonlinear electromagnetism in Cosmology.— In Ref.[6] several general properties of nonlinear electrodynamics in cosmology were reviewed by assuming that the action for the electromagnetic field is that of Maxwell with an extra term, namely [24]

$$S = \int \sqrt{-g} \left(-\frac{F}{4} + \frac{\gamma}{F} \right) d^4x, \quad (1)$$

where $F \equiv F_{\mu\nu}F^{\mu\nu}$. Physical motivations for bringing in this theory have been provided in Ref.[6]. At first, one notices that for high values of the field F , the dynamics resembles Maxwell's one except for small corrections associate to the parameter γ , while at low strengths of F it is the $1/F$ term that dominates [7]. (Clearly, this term should dramatically affect the photon- \vec{B} field interaction in intergalactic space). The consistency of this theory with observations was shown in Refs.[6, 17] using the cosmic microwave radiation bound and the anomaly in the dynamics of Pioneer spacecraft, respectively. Both provide small enough values for the parameter γ .

Therefore, the electromagnetic (EM) field described by Eq.(1) can be taken as source in Einstein equations, to obtain a toy model for the evolution of the universe which displays accelerate expansion caused when the nonlinear EM term takes over the term describing other matter fields. This NLED theory yields ordinary radiation plus a dark energy component with $w < -1$ (phantom-like dynamics). Introducing the notation [25], the EM field can act as a source for the FRW model if $\langle E_i \rangle_V = 0$, $\langle B_i \rangle_V = 0$, $\langle E_i B_j \rangle_V = 0$, $\langle E_i E_j \rangle_V = -\frac{1}{3}E^2 g_{ij}$, and $\langle B_i B_j \rangle_V = -\frac{1}{3}B^2 g_{ij}$. [26] When these conditions are fulfilled, a general nonlinear Lagrangian $L(F)$ yields the energy-momentum tensor ($L_F = dL/dF$, $L_{FF} = d^2L/dF^2$) [27]

$$\begin{aligned} \langle T_{\mu\nu} \rangle_V &= (\rho + p)v_\mu v_\nu - p g_{\mu\nu}, \\ \rho &= -L - 4E^2 L_F, \quad p = L + \frac{4}{3}(E^2 - 2B^2)L_F, \end{aligned} \quad (2)$$

Hence, when there is only a magnetic field, the fluid can be thought of as composed of ordinary radiation with $p_1 = \frac{1}{3}\rho_1$ and of another fluid with EOS $p_2 = -\frac{7}{3}\rho_2$. It is precisely this component with negative pressure that may drive accelerate expansion.

After presenting that theory in Ref.[6], we realized that there exists another *per se* equally fundamental implication of this Lagrangian. It also modifies in a significant fashion the actual redshift that one may ascribe, within a given cosmology, to a distant galaxy.

Photon dynamics in NLED: effective geometry.— Next we investigate the effects of nonlinearities in the evolution of EM waves in the vacuum permeated by background \vec{B} -fields. An EM wave is described onwards as the surface of discontinuity of the EM field. Extremizing the Lagrangian $L(F)$, with $F(A_\mu)$, with respect to the potentials A_μ yields the following field equation [9]

$$\nabla_\nu (L_F F^{\mu\nu}) = 0, \quad (3)$$

where ∇_ν defines the covariant derivative. Besides this, we have the EM field cyclic identity

$$\nabla_\nu F^{*\mu\nu} = 0 \quad \Leftrightarrow \quad F_{\mu\nu|\alpha} + F_{\alpha\mu|\nu} + F_{\nu\alpha|\mu} = 0. \quad (4)$$

Taking the discontinuities of the field Eq.(3) one gets [28]

$$L_F f_\lambda^\mu k^\lambda + 2L_{FF} F^{\alpha\beta} f_{\alpha\beta} F^{\mu\lambda} k_\lambda = 0, \quad (5)$$

which together with the discontinuity of the Bianchi identity [29] yields

$$f_{\alpha\beta} k_\gamma + f_{\gamma\alpha} k_\beta + f_{\beta\gamma} k_\alpha = 0. \quad (6)$$

A scalar relation can be obtained if we contract this equation with $k^\gamma F^{\alpha\beta}$, which yields

$$(F^{\alpha\beta} f_{\alpha\beta} g^{\mu\nu} + 2F^{\mu\lambda} f_\lambda^\nu) k_\mu k_\nu = 0. \quad (7)$$

It is straightforward to see that here we find two distinct solutions: a) when $F^{\alpha\beta} f_{\alpha\beta} = 0$, case in which such mode propagates along standard null geodesics, and b) when $F^{\alpha\beta} f_{\alpha\beta} = \chi$. In this last case, we obtain from equations (5) and (7) the propagation equation for the field discontinuities being given by [5]

$$\underbrace{\left(g^{\mu\nu} - 4 \frac{L_{FF}}{L_F} F^{\mu\alpha} F_\alpha^\nu \right)}_{\text{effective metric}} k_\mu k_\nu = 0. \quad (8)$$

This equation proves that photons propagate following a geodesic that is not that one on the background space-time $g^{\mu\nu}$, but rather they follow the *effective metric* given by Eq.(8), which depends on the background $F^{\mu\alpha}$, i. e., on the \vec{B} -field.

If one now takes the derivative of this expression, we can easily obtain [14, 15, 16]

$$k^\nu \nabla_\nu k_\alpha = 4 \left(\frac{L_{FF}}{L_F} F^{\mu\beta} F_\beta^\nu k_\mu k_\nu \right)_{|\alpha}. \quad (9)$$

This expression shows that the nonlinear Lagrangian introduces a term acting as a force that accelerates (positively or negatively) the photon along its path.

It is therefore essential to investigate what are the effects of this peculiar photon dynamics. The occurrence of this phenomenon over cosmological distance scales may have a nonnegligible effect on the physical properties that one can ascribe to a given source from astronomical observables. One example of this is the cosmological redshift, i.e., the actual shifting in the position of the absorption lines from far away galaxies. [30]. Indeed, it is proved next that as the photon travels very long distances from cosmic sources until be detected on Earth, its interaction with the background electromagnetic fields pervading intergalactic space decidedly changes the putative value of the redshift to be ascribed to the emitting source in case the NLED effects were not considered. In other words, the redshift computed in a standard fashion within a particular cosmology gets effectively affected.

Above we obtained the effective contravariant metric (see Ref.[5])

$$g_{\text{eff}}^{\mu\nu} \equiv L_F g^{\mu\nu} - 4L_{FF} F^{\mu\alpha} F_\alpha^\nu. \quad (10)$$

Since we assume that the fields detected by a comoving observer is

$$\langle F_{\mu\alpha} F_\nu^\alpha \rangle = -\frac{1}{3} B^2 h_{\mu\nu} \quad (11)$$

then, one can rewrite Eq.(10) explicitly in terms of the average magnetic field on the background as

$$g_{\text{eff}}^{\mu\nu} = L_F g^{\mu\nu} + \frac{8}{3} L_{FF} B^2 h^{\mu\nu}. \quad (12)$$

On this basis, the inverse effective metric should be

$$g_{\mu\nu}^{\text{eff}} = \frac{1}{L_F} g_{\mu\nu} - \frac{8}{3} \frac{L_{FF} B^2}{L_F (L_F + \frac{8}{3} L_{FF} B^2)} h_{\mu\nu}. \quad (13)$$

$$ds^2 = \frac{1}{L_F} dt^2 - \left(1 - \frac{8}{3} \frac{L_{FF} B^2}{[L_F + \frac{8}{3} B^2 L_{FF}]} \right) \frac{a^2(t)}{L_F} \gamma_{ij} dx^i dx^j = \frac{1}{L_F} dt^2 - \frac{1}{L_F} \left(\frac{3L_F}{3L_F + 8L_{FF} B^2} \right) a^2(t) dl^2 = 0. \quad (14)$$

It follows then that the expression for the cosmological redshift turns out to be

$$1 + \tilde{z} \equiv \frac{c\delta t_0}{c\delta t_e} = \frac{a(t_0)}{a(t_e)} \Delta = (1 + z) \Delta, \quad (15)$$

where $\Delta \equiv [(1 + \Phi)^{1/2}|_{t_e}] / [(1 + \Phi)^{1/2}|_{t_0}]$, and $\Phi \equiv 8/3(L_{FF}/L_F)B^2$. The specific modification of the redshift depends on the particular problem we focus on, in particular, a similar effect was already analyzed in the presence of very strong magnetic fields in pulsars [14, 15, 16].

As an example, in the case of cosmology, as pointed out above, a model to explain the recently discovered late acceleration of the universe using a NLED described by the Lagrangian $L(F) = -\frac{1}{4}F + \frac{\gamma}{F}$, where $F \equiv F_{\alpha\beta}F^{\alpha\beta}$ and $\gamma = -\nu^2$, was proposed in Ref.[6]. Thus, by using Eqs.(13, 15), one can compute the actual redshift of a given cosmic source.

Therefore, the cosmological redshift turns out to be

$$1 + z = \frac{c\delta t_0}{c\delta t_e} = \frac{a(t_0)}{a(t_e)} \frac{\left(\frac{B^4 + \frac{5}{3}\nu^2}{B^4 - \nu^2} \right)^{1/2} \Big|_{t_e}}{\left(\frac{B^4 + \frac{5}{3}\nu^2}{B^4 - \nu^2} \right)^{1/2} \Big|_{t_0}}. \quad (16)$$

With this equation one can plot the effective cosmological redshift according to NLED for a specific field strength B . That result is presented in Fig.1, which illustrates through the Hubble diagram, a noticeable effect on both supernovae (SNe) and afterglows of gamma-ray bursts (GRBs) redshifts. Therefore, the determination from direct observations of the redshift of a given distant quasar or galaxy leads to a mistaken interpretation on the actual distance to those sources if such NLED effect is not properly accounted for. Such a task could be performed by estimating the host-galaxy B -field.

Discussion and conclusion.— Our main result is presented in Eq.(15). Irrespective of the structure of the Lagrangian, it is valid for any generic Lagrangian $L(F)$ describing a NLED theory, provided the field averaging

That theory leads straightforwardly to prove that the cosmological redshift of a photon traveling from a distant source to Earth is also modified.

Nonlinear cosmological redshift.— With this metric one can then compute the new cosmological redshift since the line element provided by this NLED now reads

procedure indicated earlier in Eq.(11) is hold. Therefore, in order to properly address the issue on the cosmological redshift of a distant astrophysical object, this NLED effect should be first taken into account. In the specific case of the model introduced in Ref.[6] to provide an alternative to dark energy to explain the current universe acceleration phase, Eq.(15) becomes Eq.(16) which shows the dependence of the NLED effect on the relation between the host source B -field strength and the theory constant γ , as shown in Fig.-1.

As a simple exercise, one can admit for the time being that the $|_{t=0}$ (on Earth) B -field strength is much higher than the local one at the emitting source $|_{t=e}$. Since γ is a universal constant of fixed value, then, in this limit one notices that the second term in the denominator of Eq.(16) is on the order of 1. [31] By comparing to the standard calculation within a given model, viz., in Friedmann cosmology, one notices that for values of the ratio B^4/γ in the interval $(1, +\infty)$, one obtains much higher redshifts, having the standard cosmological redshift recovered in the limit $B^4 \gg \gamma$. These findings may provide a physical support to the observation-inspired interpretation that the universe is currently undergoing an accelerate expansion. Nonetheless, in any NLED theory, the actual position of the light-emitting far-away source remains unaffected! Thus, a nonnegligible affect on the claimed late-time acceleration phase appears to occur.

On the other hand, notice that for $B^4 \rightarrow |\gamma|$, the effective redshift $(1 + \tilde{z}) = a(t_0)/a(t_e) \Delta$ appears to diverge. That is, the source may appear infinitely redshifted! This seems to be a realistic possibility. Indeed it is not [32]. In fact, it is proved in a more general theory, where a term $1/F^2$ is added to Eq.(1), that this pathology definitely dissapears.[21] Aside from that, Eq.(15) makes it evident that the NLED correction is already “built-in” in the cosmic redshift estimated, for instance, from the SN host-galaxy absorption lines. In other words, after estimating the B -field strength of the host-galaxies (typically 10^8 - 10^9 G) of each of the already observed

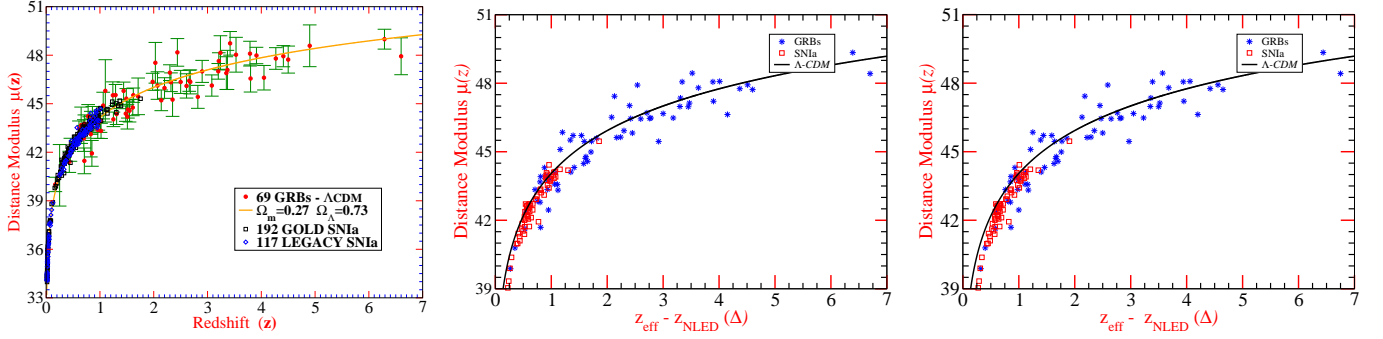


FIG. 1: Left figure: Curve (blue) delineating the NLED correction (CF_{NLED}) to the standard cosmological redshift $(1+z)$ vs. the intergalactic B -field strength normalized as B^4/γ . Right figure: Hubble diagram of the 69 GRBs sample (filled circles) of Ref.[19] calibrated for Λ -CDM, 192 GOLD SNIa (squares) and 117 LEGACY SNIa (diamonds) samples of Refs.[1, 20], as current observations indicate. One may conjecture that most, if not all, of the events presenting a much higher luminosity in this plot could have taken place in sources where the local B -field is near the critical one shown at the left. The next two graphs illustrate SNIa, GRBs global shifts of 0.1, 0.15 in z . They are intended here solely to illustrate the overall effect. The attentive reader should bear in mind that the NLED correction should indeed be applied to each individual SNIa or GRB event, once knowing (through Zeeman splitting or other techniques) the local host-galaxy B -field. This analysis will be presented elsewhere.

SNIa, the actual redshift to be plotted in the Hubble diagram ($\mu(z)$ vs. z) is going to be the effective redshift $(1+\tilde{z})$ discounting the correction factor Δ provided by NLED (examples in plots c, d in Fig.1).

From this analysis one concludes that in case the NLED theory for the photon interaction (in a vacuum) with extragalactic background magnetic fields be realized in nature, it would become evident that any conclusion on actual cosmological redshifts drawn from SNIa or/and observations in the optical of GRBs afterglows had to be revised. In any case, any general NLED theory will lead to an effective cosmic redshift $(1+\tilde{z}) = (1+z)\Delta$. In the specific case of $L(F) = F + 1/F$, wondering whether

the theory has something to do with nature rests on future experiments and/or observations like observing absorption lines in both a supernova and a gamma-ray burst afterglow in SN/GRBs related events, a connection that is by now conclusively demonstrated for a number of cases [22].

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 - [7] Notice that on dimensional grounds, $\gamma = \hbar^2 \times \nu^2$, where μ is a fundamental constant with dimension of length⁻¹.
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 - [23] In Ref.[3] has been proved that electrodynamics in a vacuum pervaded by B-fields is subject to nonlinear effects.
 - [24] Notice that this Lagrangian is gauge invariant, and that hence charge conservation is guaranteed in this theory.
 - [25] Due to the isotropy of the spatial sections of the Friedman-Robertson-Walker (FRW) model, an average procedure is needed if electromagnetic fields are to act as a source of gravity [8]. Thus a volumetric spatial average of a quantity X at the time t by $\langle X \rangle_V \equiv \lim_{V \rightarrow V_0} \frac{1}{V} \int X \sqrt{-g} d^3x$, where $V = \int \sqrt{-g} d^3x$, and V_0 is a sufficiently large time-dependent three-volume. (Here the metric sign convention $(+ - - -)$ applies).
 - [26] Let us remark that since we are assuming that $\langle B_i \rangle_V = 0$, the background magnetic fields induce no directional effects in the sky, in accordance with the symmetries of the standard cosmological model.
 - [27] Under the same assumptions, the EM field associate to Maxwell Lagrangian generates the stress-energy tensor defined by Eq.(3) but now $\rho = 3p = \frac{1}{2}(E^2 + B^2)$.
 - [28] Following Hadamard [10], the surface of discontinuity of the EM field is denoted by Σ . The field is continuous when crossing Σ , while its first derivative presents a finite discontinuity. These properties are specified as follows: $[F_{\mu\nu}]_\Sigma = 0$, $[F_{\mu\nu|\lambda}]_\Sigma = f_{\mu\nu}k_\lambda$, where the symbol $[F_{\mu\nu}]_\Sigma = \lim_{\delta \rightarrow 0^+} (J|_{\Sigma+\delta} - J|_{\Sigma-\delta})$ represents the discontinuity of the arbitrary function J through the surface Σ . The tensor $f_{\mu\nu}$ is called the discontinuity of the field, $k_\lambda = \partial_\lambda \Sigma$ is the propagation vector, and the symbols " $|$ " and " $||$ " stand for partial and covariant derivatives.
 - [29] A cyclic identity for the first derivative of the Riemann tensor, defined as: $R^\alpha_{\beta\mu\nu;\sigma} + R^\alpha_{\beta\nu\sigma;\mu} + R^\alpha_{\beta\sigma\mu;\nu} = 0$
 - [30] The same theory directly leads to a variation of the fine structure constant α [11]
 - [31] Ref.[17] has shown that $|\gamma|^{1/4} (\frac{1}{c}) = B_{\text{crit}}!$
 - [32] We caution on this apparent possibility to happen, because the procedure that was used to estimate the value of γ , and its associate B_{crit} critical field, in Ref.[17], already includes uncertainties in the average intergalactic B -field, which are very large, yet. Of course, a more consistent fashion to obtain γ would be through a dedicate laboratory experiment, as already stated in Ref.[17]. In that case, such very self-consistent method should produce a much smaller value of γ than the one computed and described in footnote-3 of Ref.[17], so that the chance of divergence disappears.